Teaching learning projects and didactical engineering

Colette Laborde
Cabrilog

Abstract. *Since a long time, long-term teaching projects have been developed in several countries for improving the learning of mathematics. In the eighties, the concept of didactical engineering emerged in France as a method of designing and evaluating teaching learning projects. This paper examines the historical roots of this method and presents its specific features. It also evokes the extension and transformation of this method in a more recent past.*

Keywords: teaching learning projects, didactical engineering, dialectic between theory and empirical investigations, problem situations.

Sunto. *Da tempo, in diversi Paesi sono stati sviluppati progetti di insegnamento a lungo termine per migliorare l'apprendimento della matematica. Negli anni ottanta, il concetto di ingegneria didattica è emerso in Francia come un metodo per la progettazione e valutazione di progetti di insegnamento-apprendimento. Questo articolo esamina le radici storiche di questo metodo e presenta le sue caratteristiche specifiche. Vengono anche evocate l'estensione e la trasformazione di questo metodo in un passato più recente.*

Parole chiave: progetti di insegnamento-apprendimento, ingegneria didattica, dialettica fra teoria e ricerche empiriche, situazioni problema.

Resumen. *Desde hace tiempo, en diversos países se desarrollaron proyectos de enseñanza a largo termino para mejorar el aprendizaje de la matemática. En los años ochenta, el concepto de ingeniería didáctica emergió en Francia como un método para el diseño y la evaluación de proyectos de enseñanza-aprendizaje. Este artículo examina las raíces históricas de este método y presenta sus características específicas. Son también evocadas la extensión y la transformación de este método en un pasado más reciente.*

Palabras clave: proyectos de enseñanza-aprendizaje, ingeniería didáctica, dialéctica entre teoría e investigaciones empíricas, situaciones problemas.

1. Introduction

For many years, mathematics curricula as well as numerous innovative sometimes long-term teaching projects in mathematics education have been designed with the intention to improve the learning of mathematics by students. Such projects occurred in several countries long before a specific scientific research in didactics of mathematics was established.

At the beginning of the 80's, after about 10 years of implementing scientific research in didactics, the French community of research introduced the idea of didactical engineering. Some researchers as Chevallard (Artigue, 1990) urged the community to eventually cope with theorizing the critical and complex real object of didactics of mathematics, i.e. the actual functioning of the didactical system, or in other terms the functioning of teaching sequences in classrooms with real students and real teachers.

This idea of didactical engineering introduced a new dimension, with respect to the attempts of designing curricula and innovative teaching projects, a dialectical relationship between theory and practice: "Con questa dizione si intende lo studio condotto in modo scientifico (quanto meno razionale) del fenomeno didattico; la messa in evidenza di una realizzazione didattica concreta, come attività di ricerca per verificare le costruzioni teoriche" (D'Amore, 1999, p. 228). As such, the method of didactical engineering may seem to constitute a break with the past design of teaching projects. However, its roots can be found in the reforms of the French teaching of mathematics from the very beginning of the XX century. This paper intends to present these roots and to show how seeds of what constitutes the essence of didactical engineering can be found in projects developed in France before the official birth of this concept. It also sketches how didactical engineering changed over the time under the questions coming from its use.

2. The New Math reform in France

France shares with Italy the fact that the teaching of mathematics and the design of curricula in mathematics have drawn the attention of many members of the *noosphere*, "the 'sphere' of those who 'think' about teaching (...) who share an interest in the teaching system, and who 'act out' their impulses in some way or another" (Chevallard, 1991). Those members in France were from various origins: university mathematicians, leaders of the association of mathematics teachers (APMEP, Association des Professeurs de Mathématiques de l'Enseignement Public), persons in charge of controlling the content to be taught like state inspectors of mathematics teaching and, after 1969, members of the IREMs (Instituts de Recherche sur l'Enseignement des Mathématiques, created after 1968 in particular at the strong request of the mathematics teacher association).

Among these members of the noosphere, mathematicians played a decisive role. Artigue (1998) gives two examples of reforms done in France at the initiative of mathematicians: the reform of 1902 and the New Math reform that affected also a wide range of countries across the world, in particular Italy:

> L'Italie a été l'un des pays d'Europe dans lesquels la connaissance des orientations de renouvellement de la vision culturelle des mathématiques soutenues par Choquet, Dieudonné et Lichnerowicz s'est répandue en profondeur dès les années 50 dans les milieux académiques et parmi les enseignants les plus

engagés. (Boero, 1994, p. 18)

In the ’50s the mathematics program of studies for middle and secondary schools (enseignement secondaire, 11- to 18-year-old students) appeared in several countries as obsolete and inadequate with respect to the scientific and technical progress of the society, according to members of the noosphere from various origins.

The mathematicians involved in the reform were probably also influenced by the prominent place of structuralism in various domains of scholarly knowledge, such as psychology, linguistics, and of course mathematics, deeply marked by the French Bourbaki movement. The same strong hypothesis about learning prevailed in France and in Italy: The contents of teaching could not give a chance to students to really understand them because they were mathematically unfounded.

The mathematical content to be taught was deeply changed under the umbrella of structuralism without really taking into consideration its impact on the actual everyday teaching in classroom and on learning processes. In the case of geometry, a new organization was proposed in particular by the French mathematician Choquet (1964), aiming at finding the best set of axioms for a presentation providing a logic sequencing of the contents appropriate for secondary mathematics. In particular, a choice had to be done between a light system of axioms and a heavier set. A heavier set avoids a long and tedious path to theorems whereas a light system minimized what had to be accepted. Such a proposal of reorganization of geometry was based primarily only on the block of knowledge in terms of a praxeological approach. It was proposed as an example for future curricula and certainly affected the design of the New Math program of studies in France. One can recognize in this enterprise the concentration on the pure mathematical content and the wish to give a better foundation to the mathematical contents before embedding them into a curriculum.

3. The reactions to the New Math reform

However, simultaneously to the structuralist movement, other ideas emerged, in particular through the CIEAEM (Commission Internationale pour l’Etude et l’Amélioration de l’Enseignement des Mathématiques). Founded by Caleb Gattegno, it managed to mobilize people from various background, as for example, the psychologist Piaget, the philosopher Gonseth, the mathematicians Dieudonné, Lichnerowicz, Choquet, and the teachers Emma Castelnuovo, Lucienne Félix, and Willy Servais (Furinghetti, Menghini, Arzarello, & Giacardi, 2008). Actually, the CIEAEM had a twofold influence on the teaching of mathematics:

- the one favoring the structuralism, expressed in CIEAEM first book

(Piaget et al., 1955) by Dieudonné, Lichnerowicz, Choquet, and Piaget, who linked the structures of the mind with mathematical structures;

- and the other more focused on the practice in classroom and on the use of concrete and semi-concrete material, as expressed in the second book of the commission (Gattegno et al., 1958).

According to Furinghetti et al. (2008) this second book "comprises educational ideas and features that are still under discussion: computer, embodied cognition, gestures". (About embodied cognition and gestures, see: Arzarello & Robutti, 2008; Arzarello, Paola, Robutti, & Sabena, 2009). One may also read a recent interesting discussion of some aspects introduced by CIEAEM in D'Amore and Fandiño Pinilla (2017) in which they review some illusory methodological proposals for mathematics education done in the past.

Some mathematicians also reacted. Freudenthal (1958, p. 7) criticized the "anti-didactic inversion introduced by the axiomatic in geometry" (Furinghetti et al., 2008) and supported the work done in geometry by Mr. and Mrs. Van Hiele in Nederland. Later, the French mathematician Thom (1973) claimed that the focus should be more on meaning than on rigor and that eliminating geometry in favor of algebra has eliminated the link between natural language and abstraction. Some good projects for renewing the mathematics teaching were carried out in Italy under the guidance of mathematicians, who were interested in mathematics teaching (Furinghetti, 2006).

In France, at the end of the '60s and in the '70s, the French association of the mathematics teachers (APMEP) reacted by proposing a deep change in schools avoiding abstraction in the teaching. The commitment of its members in the everyday teaching led the leaders of the association not to be satisfied by the mere change of the contents to be taught. They wanted to radically change the teaching methods and to leave the world of principles for definitely setting up practical modalities of action in the classrooms (de Cointet, 1975). The association advocated for a change of the program of studies. The programs should contain:

- a list of core subjects consisting of the concepts to be acquired by students in the school year;
- a list of themes to be chosen by the teacher for either motivating the introduction of new knowledge, or illustrating the usefulness of already introduced knowledge, or nurturing new and free investigations. In particular applications related to real life were considered as providing fruitful themes.

The need for aligning mathematics with reality and the modelling role of mathematics may be linked with several Italian teaching projects that aim at constructing mathematical knowledge through contextualization in domains of experience outside mathematics (Boero, 1994). The book by Castelnuovo and Barra (1976), *Matematica nella realtà*, is a representative of this trend.

Members of the association and the IREMs undertook a huge work on the links between core knowledge and themes. Many innovative proposals of teaching by means of core themes (noyaux-thèmes) were published in the journal of the association, in booklets or in textbooks (see, in particular, *Le Bulletin Vert* de l'APMEP, n° 300, September 1975).

The failure of the New Math reform provided evidence of the need of taking into account more than the mathematical contents, to make use of some general pedagogical and psychological principles (Artigue, 1990, 1998) in the search for improving teaching, and to know more about the learning processes and the links between teaching and learning. This certainly contributed to the need of developing research on mathematics teaching and learning that included also empirical elements. The same need emerged at the international level as expressed by Begle (1969) in his lecture at the first ICME in Lyon: Neither teachers, nor mathematicians, nor mathematics educators had "been in a position to gather, during the course of our ordinary activities, the kind of broad knowledge about mathematics education we need" (p. 239).

Some of the innovations taking place in French schools during the '70s were forerunners of the method of didactical engineering.

4. Teaching projects based on a dialectics between theory and empirical investigations

In France, long-term teaching projects covering almost all the mathematics teaching about numbers and measurement were designed and experimented in primary school by Brousseau, Douady, and Perrin-Glorian, from the early seventies. From an institutional point of view, French primary school could be more easily a place for long-term experimentations than secondary school. The culture in primary school differed from the one in secondary schools. In particular primary school teacher education was not carried out in universities at that time.

The projects mentioned above started from mathematical choices. In Brousseau's project about decimal numbers (1997, chapters 3 and 4), the epistemological rationale is to introduce decimal numbers as economical tools through which comparing, adding, and subtracting fractions can be done more quickly and with fewer errors. In particular, some types of problems – such as finding a new fraction lying between two given fractions – could also be solved more easily. The main idea of the teaching process involves constructing rational numbers as tools for measuring, and then decimal numbers as tools for approximating rational numbers. The final part of the teaching sequence focuses on rational numbers as operators, culminating in construction of the product of two rational numbers in terms of the composition of two mappings. In the project of Douady and Perrin-Glorian (Douady, 1980, 1986), decimals were introduced as approximating real numbers in measures of lengths that were supposed to be concepts known to the students.

However, although the mathematical choices were prevailing in these projects, their design contained in action some theoretical aspects that were later formulated and theorized by their authors. For example, a lever used by Douady in the sequencing of the tasks was the interplay between the numerical and the geometrical settings on the one hand, and between settings and registers on the other hand. The notion of setting introduced by Douady refers to a set of objects and relationships between them belonging to a domain of mathematics. A setting not only includes objects and relationships but also various formulations and mental images. Examples of settings are the numerical setting, the geometrical setting, the algebraic setting. The term *register* denotes here a semiotic register, i.e. a semiotic system for representing objects and relationships (Duval, 2006). The role of visual representation was not only to express and support mathematical thinking. It was also meant for posing problems that cannot be solved in the register in which it was expressed, in order to require a move to another setting and/or register.

One of the problems posed in the teaching sequence by Douady for introducing decimal numbers was to find the length of the side of a square with given area equal to a whole number. The 8- to 9-year-old students could not solve the problem in the numerical setting. The graphical register with a system of axes was introduced. Students had to represent a rectangle with dimensions a and b by means of a point (a, b) on this system. A whole number p was given. Students had to represent many rectangles then to color each obtained point in red if the area of the corresponding rectangle was larger than p, in blue if it was less than p and in black if it was equal to p. Then they had to find points representing rectangles with area equal to p. The initial question became: Is there a square among the rectangles and what is the length of its sides?

In the design of the teaching project about decimals, Brousseau examined how decimal numbers had evolved within the wider field of mathematics in order to identify the key mathematical problems which gave rise to decimal numbers, and to clarify the relationships between decimal numbers and other types of number, especially rational numbers, typically expressed in the form a/b. He also investigated the former and current presentations of decimals in teaching. These studies were published later (available in English in Brousseau, 1997, chapter 3). Brousseau started from the notion of obstacle proposed by Bachelard and distinguished between three origins of obstacles: ontogenetic, epistemological, or didactical. After Bachelard (1938), who introduced the notion of epistemological obstacle, i.e. an obstacle constitutive of the way of knowing, Brousseau (1997) considered that knowledge is simultaneously support and obstacle. Obstacles are made apparent by errors or inefficient processes. Such errors and processes are not due to chance but are persistent and reproducible. Knowledge is very efficient in certain situations but can be inappropriate for other ones. Whereas epistemological obstacles can be found in the history of the concepts themselves, didactical obstacles stem from the presentation of a concept and the way of using it in teaching. Inherited from a long tradition, a widespread presentation of decimal numbers is associated to measurement and related to technical operations on whole numbers. “As a result, for students today, decimal numbers are whole numbers with a change of units” (Brousseau, 1997, p. 87). Overcoming obstacles means transforming knowledge acquired by the learner. Facing the learners with problem situations and organized *milieux* with which the learner interacts is the way proposed by Brousseau.

The authors of these two long-term projects did not claim to have used a didactical engineering method because the term was not yet coined. However, it is very clear that problems or rather problem situations to which the students were faced played a fundamental role for creating the conditions of students’ development and transformation of knowledge.

5. Problem situations and development of students’ knowledge

In the mid-seventies, after the New Math reform, research on the development of students’ conceptions and understandings took place in France simultaneously to the design and experimentation of these long teaching sequences.

Feeling that it was not enough to change the curricula without knowing more about the ways students understood mathematical concepts, several researchers investigated the solving strategies and erroneous procedures of students faced with well-chosen problems in order to propose models of their thinking processes, understandings, and conceptualization (Vergnaud, 1991).

Student conceptions of specific mathematical notions could be identified through situations students were faced with, as expressed by Rouchier (1980) and Artigue and Robinet (1982). When presenting conceptions about the notion of circle at primary school, Artigue and Robinet wrote that they did not want to analyze the students’ conceptions independently of a precise study of situations in which these conceptions were involved.

Situations are chosen according to the conceptions they may favor and, if they are in a sequence, their order is also chosen in the same way. Several research projects studied how students’ knowledge developed in a sequence of problem situations carried out in classroom with teachers’ interventions and collective phases under the guidance of the teacher.

Two significant examples are given by the situations and didactical processes on rationale positive numbers (Rouchier, 1980) and the didactical experiment on the concept of volume (Vergnaud et al., 1983). This latter research project gave rise to a whole issue of the newly created French journal, *Recherches en didactique des mathématiques*, which consisted of three articles: the first one on the conceptions and competences of students of four middle school classes when faced with tasks outside the classroom (Ricco, Vergnaud, & Rouchier, 1983), the second one on a sequence of didactical situations done in a grade 7 classroom (Vergnaud et al., 1983), and the third one on a comparison between students' answers to a questionnaire given before the sequence and to the same questionnaire given after the sequence (Rogalski, Samurçay, & Ricco, 1983). In the introduction of the issue (pp. 23–24), Vergnaud claimed how the theory of situations, the psycho-genetic complexity and the task analyses complement each other. However, his argument reveals that the general aim of the study lies in investigating the genesis of knowledge on a short term for the teaching sequence and on a longer term for the interviews:

> Il existe un temps long de la psychogenèse, bien connu des psychologues, qui se mesure en années et qui permet d'établir des hiérarchies dans la complexité des problèmes et des concepts mathématiques. Il existe aussi un temps court de la psychogenèse, moins bien étudié que le premier et pourtant essentiel en didactique, qui concerne l'évolution des conceptions et des pratiques d'un sujet ou d'un groupe de sujets face à une situation nouvelle. (p. 24) [There is a long-term time of the psycho genesis, well known from psychologists, that is measured in years and allows to establish hierarchies in the complexity of problems and mathematical concepts. There is also a short-term time of the psychogenesis, less studied than the former one but essential in didactics that deals with the development of conceptions and practices of an individual or a group of individuals faced with a new situation].

6. Didactical engineering

At the time of these investigations, i.e. at the beginning of the eighties, the term *didactical engineering* appeared in articles and internal meetings of the French community of researchers in mathematics education (Artigue, 1994). Didactical engineering refers to a method that aims at carrying out empirical studies of didactical phenomena in circumstances compatible with an ethical study of teaching, i.e. in the real and complex setting of classrooms:

> Artigue (…) describes didactical engineering as similar to the work of the engineer, who is acquainted with the major scientific knowledge and accepts the scientific methods but at the same time is obliged to work with very complex objects, far from the simplified objects which are studied by science. (Margolinas & Drijvers, 2015, p. 897)

A didactical engineering consisted of four phases: design of a teaching sequence made of a sequence of situations, experimentation in one or several classrooms, observation of the students' activity and of the teacher's interventions as well as of the collective discussions, analysis of the observations.

It becomes a method and is no longer an innovation as soon as the design of the situations considers each situation as depending on global and/or local variables. A variable of a situation (or of a task) is a feature of this situation affecting the possible solving strategies. Playing on such a feature may make the task easier or more difficult. It is a lever in the hands of the teacher or the designer of the tasks in order to foster the construction of knowledge by the learner. For example, in an additive task, the nature of the numbers is a variable that can have different values such as integers, decimals, fractions. The task is easier with whole integers than with decimals or with fractions – and often easier with decimals than with fractions.

For each situation, the researcher analyses the possible effect of different values of the variables on students' solving strategies and chooses the values according to the strategies (s)he wants to favor. Each situation is not considered isolated from the other ones but within the whole sequence of situations. Values of variables are chosen in order to foster an expected development of students' strategies during the sequence. Two components of the method are critical:

- the design of situations;
- the a priori analysis and the internal validation.

6.1. The design and role of situations

A keystone is indeed the notion of *situation* calling for a specific functioning of knowledge. The problem is the source and criterion of mathematical knowledge from both epistemological and cognitive perspectives, wrote Vergnaud (1981) who later preferred to replace the word *problem* by *situation* under the influence of the theory of didactical situations by Brousseau. Piaget's theory of *equilibration* (Piaget, 1975) was a crucial source for the idea of adaptation in which students construct new knowledge through becoming directly engaged in solving a novel type of problem, refining their concepts and strategies in the light of feedback from a material and social milieu (Brousseau, 1997, pp. 64, 147). Here *situation* refers to a collection of problem-solving tasks and task environments designed to evoke a particular form of *a-didactical* adaptation on the part of students, and intended to help them construct some specific new knowledge. The adjective *a-didactical* refers to the fact that the students must experience the task not as intended to teach them but as if they had to cope with a real problematic situation outside the classroom and find a way to solve it with all their means.

Designing a situation not only means designing a problem but also determining the conditions under which it will be solved, the means of action of the students and the feedback they will receive from the environment in the solving process. Conditions, means of actions, and feedback depend on variables on which researchers can play for favoring an expected development of students' strategies.

In geometry, the move from a property used in action by students to another one less familiar can also be organized by a play on available instruments. A good example is given by the didactical engineering on reflection at grade 6 proposed by Grenier (1990). Paper folding is given at first for introducing symmetry lines, but then paper folding plays only the role of empirical checking of the validity of a construction. Students are rapidly asked to construct symmetry lines of figures without resorting to paper folding and with specific instruments in order to favor the use of mathematical properties. The play on instruments is systematically used to hinder the use of certain properties and favor the use of other ones.

For example, drawing the symmetry line of an isosceles trapezoid with only a straightedge and a set square cannot be done by using the midpoints of the parallel sides of the trapezoid but by using intersection point of diagonals *BD* and *AC* or lines supporting sides *AD* and *BC* and/or perpendicularity of the symmetry line and the parallel sides of the trapezoid (Figure 1). Only later the teacher was supposed to formulate the properties used in action by students, after she has gathered the various strategies.

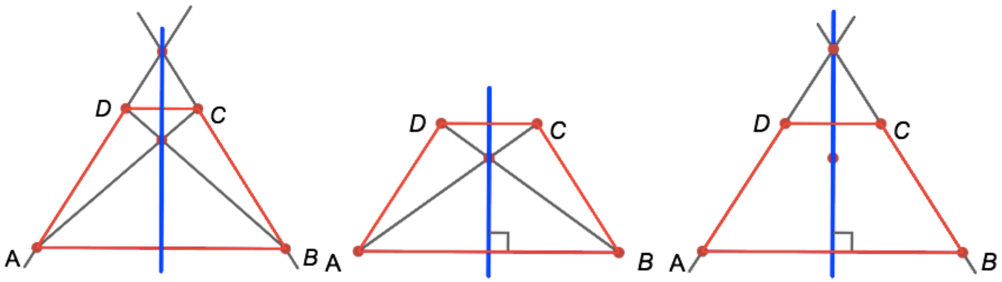

Figure 1. Constructing the symmetry line of an isosceles trapezium without using midpoints.

Didactical engineering gives an important place to problems and the organized milieu. The construction of these problems and of a milieu is done by means of an a priori analysis.

6.2. A priori analysis *and internal validation*

The design of the teaching sequence in didactical engineering is based on an *a priori analysis* that plays a critical role since the a priori analysis is contrasted with the a posteriori analysis of the observations of the implementation in the classroom. *A priori* does not refer to a temporal place (i.e. prior to the experimentation) but refers to the independence of the analysis from any empirical fact coming from the experimentation. The discrepancies between a priori and a posteriori analyses lead to reconsider the hypotheses on which the a priori analysis is based and allow refining or even modifying the theoretical approach underpinning the research work.

As explained above in the example of the teaching decimals project by Brousseau, the a priori analysis devotes a large part to the epistemological analysis of the mathematical content involved in the teaching project, but also to a cognitive analysis of the available knowledge of the students based on research results, and to an analysis linked to the institutional functioning of the teaching. These two latter analyses are critical for the functioning of the situations in classrooms. The a priori analysis must take into account the teaching constraints coming from the program of studies, as well as the possibility for the teacher to manage the project in the classroom. At the cognitive level, an a-didactical situation is expected to foster the emergence of new solving procedures which will be the seed of new knowledge. Students must be able to start solving the task of this situation, but with an incomplete or tedious procedure if the a-didactical situation cannot play its role. The design of such situations must optimize the choice of the variables of the situation in order to secure as much as possible the expected processes of the students and the adequacy of the teaching project with the usual teaching in the classroom.

Whereas the projects mentioned above on rational positive numbers and on the concept of volume used pre- and post-questionnaires to assess the learning by students, the didactical engineering proceeds by using an internal validation. Validation is done by comparing the students' expected solving processes in the a priori analysis and the observed processes in the classrooms. In case of discrepancies, an analysis of the students' solving processes is carried out and may lead to modify the expected role of the variables of the situations or reveal elements of the milieu not taken into account in the a priori analysis, as in the teaching sequence by Grenier (1990, section 6).

The internal validation process is a characteristic feature of didactical engineering and makes it different from other types of teaching experiments in classrooms.

7. Extension of didactical engineering

In a first period of time, didactical engineering investigated the teaching of specific concepts or, as said above, the development of students' conceptions in a sequence of problems, generally at primary or secondary school.

Later the method sheds light to components of the teaching process that were not enough investigated and theorized. Finally, it was used for studying general didactical phenomena. Let us give some examples.

Grenier (1990) experimented a first time a teaching sequence on reflection at grade 6. Contrasting the a priori analysis with the a posteriori analysis, she observed that the play on instruments did not necessarily lead to a change of solving strategies. When they did not have measurement tools, the students tried to estimate measures by eye or using a pen as a measurement unit, instead of using geometrical properties. The interventions of the teacher seemed to have no effect on students' strategies. Grenier modified the situations for another teaching experiment the following year, but even if the trajectories of the students were closer to the expected ones, the analysis of the observations revealed strong resistances both in the students' conceptions and in the teacher interventions. In collective debriefings of the group work, the teacher ignored some popular strategies and focused on strategies used by a small number of students because they were the expected ones. He rejected strategies of measurement with a pen or with the section of a ruler, by saying that it was not precise enough. This argument was not understood by students who thought that using a measurement was more precise than using the fact that points are collinear. This research showed that the a priori analysis could not deal only with situations but also with teachers' interventions and decisions. The a priori analysis had also to take into account phenomena related to the didactical contract. Some behaviors of students and teachers can be explained only by the fact that there are implicit rules underlying the progress of the classroom. This research showed very clearly how much a teaching sequence results from a balance between two poles: the a-didactical pole and the pole related to the didactical contract (Brousseau, 1990).

Didactical engineering was used at the tertiary level (Robert, 1992; Dorier, Robert, Robinet, & Rogalski, 1994) and questioned the construction of knowledge as a tool for solving problems at that level. More than efficient tools for solving a class of problems, concepts taught at the tertiary level own a power of generalization and unification of different strategies and methods. It seems difficult that students can construct such concepts on their own from a-didactical situations.

The study of phenomena related to the integration of technology into the teaching of mathematics used the didactical engineering method. For example, instrumentation processes of dynamic geometry were investigated by Restrepo (2009) in a long-term didactical engineering (one year) method.

The robustness of the didactical engineering method was also investigated by using teaching sequences designed with a didactical engineering method in other conditions. For example, Perrin-Glorian (1993) showed that it is very difficult for less advanced students to engage in a-didactical situations and produce new solving strategies. It was also very difficult for them to understand the institutionalization phase done by the teacher. In this phase, the teacher extracted and formulated the mathematical and official knowledge from a-didactical situations. The students did not understand the link between the teachers' discourse and what they experienced in the situations.

Especially over the twenty past years, the landscape of research in mathematics education changed a lot. Two phenomena must be mentioned:

- the only indirect influence of didactical engineering on teachers everyday practice and the move to second generation didactical engineering (Perrin-Glorian, 2011);
- the growing use of networking of theoretical frameworks (Artigue, 2009a) at a national level as well as the international level.

It turns out that teachers do not make use of original situations of didactical engineering, but instead use simplified and isolated situations presented in worksheets, with the main ideas originating from situations developed in didactical engineering (Perrin-Glorian, 2011). This resonates with Bartolini Bussi's (2005) regret of the too heavy weight of the theoretical considerations of the papers of an *Educational Studies* special issue about French research on classroom situations: "It may act as an obstacle to the diffusion of results and methods" (p. 305). Perrin-Glorian investigated the transformation process of an original didactical engineering into a didactical engineering appropriate for teaching and claims that this transformation requires work, in particular on the conditions of the transmission of the engineering.

The internalization of research, as well as the complexity of the processes in mathematics teaching, led to use several theoretical frameworks in a same research project. Whereas didactical engineering initially was developed within the theory of didactical situations, the method was then attached to other theoretical frameworks. For example, the instrumentation theory was associated to the theory of didactical situations or with the anthropological theory of didactics to study the use of digital technology in mathematics teaching (Artigue, 2009b). Frameworks developed outside of France shed light on aspects of the teaching process less addressed by the first French didactical engineering projects. An appropriate example is provided by the theory of semiotic mediation (Bartolini Bussi & Marriotti, 2008) which focuses its attention on the role of signs in the construction of knowledge and on the transformation processes of personal signs constructed by students into mathematical signs. Teaching projects elaborated within the theory of semiotic mediation give a central role to the semiotic action of the teacher, in particular in the collective discussions she organizes and orchestrates, and in which she bridges personal meanings of the students constructed in their activities to mathematical meanings. For example, Mariotti (2013) designed long-term teaching projects making use of the semiotic potential of software programs: Students are faced with purposefully designed tasks within computer environments fostering the development of personal meanings that the teacher helps evolve through carefully conducted collective discussions. The method used in these teaching sequences consist of analyzing the formulations of the teacher and of the students in order to identify signs and their transformations as well as the discursive strategies of the teacher. It is clear that sophisticated analyses of teachers' discursive strategies cannot result from an a priori analysis and the authors of these projects may not refer to the method of didactical engineering even if the design of tasks given to students is carefully chosen with learning aims in mind.

The long-lasting links between the Italian and French communities of mathematics education led to teaching learning projects resorting to frameworks from both countries making an original use of didactical engineering. For example, the theories of didactical situations and of semiotic mediation were used in interaction in a long-term teaching project about graphs of functions (Falcade, Laborde, & Mariotti, 2007). The a posteriori analyses showed how the same observed phenomenon can be interpreted differently in each framework and thus lead to establish bridges between both frameworks. The power of a cross-analysis methodology also resorting to the theory of didactical situations and to the theory of semiotic mediation is very well exemplified in Maracci, Cazes, Vandebrouck, and Mariotti (2013).

The history of didactical engineering showed that concerns about the contents to be taught at the time of the New Math reform provided a context for investigating teaching and learning phenomena beyond the pure mathematical content. The method of didactical engineering started as a method for better understanding the relationships between the design of problem situations and the development of specific mathematical concepts by students. The method was then extended into several directions: length of the teaching experiment, teaching at the tertiary level, less advanced students, use of technology, studies of the didactical contract and of the role of the teacher, and finally led to study other phenomena related to teaching.

Didactical engineering is still a research method. The development of resources for mathematics teachers is nowadays becoming a critical issue with the increasing number of resources available on Internet. What are the best ways of transmitting didactical engineering products in order to facilitate their use by a large number of teachers without changing their impact on the learning processes? (Perrin-Glorian, 2011). What are the conditions for such a didactical engineering product to be really used in ordinary teacher practice? Which mathematical and didactical knowledge do the teachers need to make use of such resources? Many questions remain and renew the research questions related to didactical engineering.

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